Received: *March 23, 2001*, Accepted: *June 5, 2001*
Revised: *May 15, 2001*

Noncommutative solitons in open $N = 2$ string theory

Olaf Lechtenfeld, Alexander D. Popov[*] and Bernd Spendig

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
Email: `lechtenf@itp.uni-hannover.de`, `popov@itp.uni-hannover.de`, `spendig@itp.uni-hannover.de`

Abstract: Coincident D2-branes in open $N = 2$ fermionic string theory with a B-field background yield an integrable modified $\mathrm{U}(n)$ sigma model on noncommutative $\mathbb{R}^{2,1}$. This model provides a showcase for an established method (the 'dressing approach') to generate solutions for integrable field equations, even in the noncommutative case. We demonstrate the technique by constructing moving U(1) and U(2) solitons and by computing their energies. It is outlined how to derive multi-soliton configurations with arbitrary relative motion; they correspond to D0-branes moving inside the D2-branes.

Keywords: Solitons Monopoles and Instantons, D-branes, Non-Commutative Geometry, Integrable Field Theories.

[*]On leave from Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Russia

Contents

JHEP06(2001)011

1. Introduction and results

Solitonic solutions of field equations of motion play an essential role in our understanding of field and string theories beyond perturbation theory. This persists for the noncommutative extension of scalar and gauge field theories which have lately come under intense scrutiny because they appear naturally in string theory [1]–[5]. Indeed, many quantum field theories on a noncommutative space-time are obtained in a certain zero-slope limit from effective field theories of open-string modes in a particular NS-NS two-form ('B-field') background [6]. In modern parlance, open-string dynamics takes place inside the world volume of D-branes where a constant B-field gets traded for effective noncommutativity and a deformed metric. Hence, string theories provide a higher-level (commutative) arena for the study of noncommutative field theories.

A case in point is the open $N = 2$ fermionic string [7, 8, 9] which at tree-level[1] proved to be identical to self-dual Yang-Mills theory in $2 + 2$ dimensions, possibly in the background of a self-dual metric and a Kähler B-field [11]. Turning on the latter in the world volume of n coincident (kleinian) D3-branes was recently shown to yield noncommutative self-dual U(n) gauge theory [12].

Starting from D3-branes whose world volumes fill the flat (kleinian) target space $\mathbb{R}^{2,2}$, it is interesting to consider lower-dimensional D-branes and describe the effective field theories appearing on their world volumes in a zero-slope limit. In this paper

[1] For a one-loop analysis see [10].

we show that open $N = 2$ strings in a B-field background induce on the world volume of n coincident D2-branes a noncommutative generalization of a modified $\mathrm{U}(n)$ sigma model (with a Wess-Zumino-type term) [13]. The latter is equivalent to non-relativistic Chern-Simons theory coupled to Higgs fields [14]. For n coincident D1-branes we obtain the standard $\mathrm{U}(n)$ sigma model in $1+1$ dimensions and its euclidean version.

By now a lot of solitonic solutions for noncommutative Yang-Mills and Yang-Mills-Higgs theories have been constructed [15]–[37], many of them with the help of the solution-generating technique proposed in [28]. In the present paper we demonstrate that one can do better for '*integrable*' noncommutative field theories. Whenever equations of motion can be written as compatibility conditions of some *linear* equations, then there exists a more general technique to construct a new solution from an old one, known as the '*dressing*' approach [38, 39, 40].

We adapt this dressing method to the noncommutative case and apply it to the noncommutative modified sigma model in $2+1$ dimensions (mentioned above). It is outlined how to construct *multi*-soliton configurations with arbitrary *relative motion* of the individual lumps. To be explicit we illustrate the general scheme by giving examples of one-soliton solutions for the gauge groups U(1) and U(2), including their energies. Since the model has neither relativistic nor rotational invariance, the energy of a noncommutative soliton depends on the direction of its motion. For some directions it even falls *below* the rest energy. If all velocities are put to zero one simply obtains static configurations which are the instantons of the noncommutative two-dimensional sigma model. In general, though, our solitons correspond on the string level to D0-branes moving inside D2-branes.

2. Open $N = 2$ strings in a B-field background

We begin by recapitulating some basic facts about the open $N = 2$ string and how to properly couple it to a B-field background. The critical $N = 2$ string lives in $2+2$ (real) dimensions, i.e. the string world sheet Σ is embedded into a four-dimensional target space with (kleinian) signature $(2,2)$. Its propagating world-sheet degrees of freedom consist of the embedding coordinates X^μ (with center-of-mass x^μ) plus a set of NSR Majorana spinors ψ^μ, with $\mu = 1,2,3,4$. The $N = 2$ string admits couplings to a self-dual metric $g = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$ and a Kähler NS-NS two-form $B = B_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. We specialize to flat kleinian space $\mathbb{R}^{2,2}$ with a metric

$$(g_{\mu\nu}) = \mathrm{diag}(\xi_1, \xi_1, -\xi_2, -\xi_2)\,, \qquad \text{where } \xi_1, \xi_2 \in \mathbb{R}_+ \tag{2.1}$$

are scaling parameters. In addition, we shall consider n coincident Dp-branes for $p = 1,2,3$ and gauge to zero all B-field components orthogonal to the D-brane world volumes. Concrete expressions for the remaining B-field components shall be given in the next section.

It has been shown [41, 12] that the standard superfield action in superconformal gauge has to be improved by two B-dependent boundary terms, in order to achieve consistency of the open-string boundary conditions with the two residual rigid supersymmetries [42, 43][2]

$$\begin{aligned} \delta X^\mu &= \overline{\varepsilon}_1 \psi^\mu + J^\mu_\nu \overline{\varepsilon}_2 \psi^\nu \,, \\ \delta \psi^\mu &= -i\rho^\alpha \partial_\alpha X^\mu \varepsilon_1 + i J^\mu_\nu \rho^\alpha \partial_\alpha X^\nu \varepsilon_2 \,. \end{aligned} \tag{2.2}$$

Here (J^μ_ν) is a constant complex structure compatible with our metric and B-field, i.e.

$$g_{\mu\nu} J^\nu_\lambda + J^\nu_\mu g_{\lambda\nu} = 0 \qquad \text{and} \qquad B_{\mu\nu} J^\nu_\lambda - J^\nu_\mu B_{\lambda\nu} = 0 \,. \tag{2.3}$$

Its nonzero components are the following: $J^1_2 = -J^2_1 = J^3_4 = -J^4_3 = 1$. The consistent gauge-fixed $N=2$ string action reads [12]

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left[(\eta^{\alpha\beta} g_{\mu\nu} + \varepsilon^{\alpha\beta} 2\pi\alpha' B_{\mu\nu}) \partial_\alpha X^\mu \partial_\beta X^\nu + (g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}) i \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi^\nu \right] . \tag{2.4}$$

It is important to note that the action functional (2.4) cannot be written in terms of superfields.

3. Branes and Seiberg-Witten limits

General formulae. We now investigate how open $N=2$ strings are affected by a background B-field on D-brane world volumes. Without loss of generality, we only turn on B-field components parallel to the D-brane world volumes. Furthermore, we switch on only their 'magnetic' components, in order to allow for the Seiberg-Witten limit to yield noncommutative gauge theory.

The starting point is the form of the open-string correlators [4, 6],

$$\begin{aligned} \langle X^\mu(\tau)\, X^\nu(\tau') \rangle &= -\alpha' G^{\mu\nu} \, \ln(\tau - \tau')^2 + \frac{i}{2} \theta^{\mu\nu} \, \varepsilon(\tau - \tau') \,, \\ \langle \psi^\mu(\tau) \psi^\nu(\tau') \rangle &= \frac{G^{\mu\nu}}{\tau - \tau'} \,, \end{aligned} \tag{3.1}$$

for $\tau, \tau' \in \partial\Sigma$. Here,

$$[(g + 2\pi\alpha' B)^{-1}]^{\mu\nu} = G^{\mu\nu} + \frac{\theta^{\mu\nu}}{2\pi\alpha'} \tag{3.2}$$

yields the effective metric $G = (G_{\mu\nu})$ seen by the open string and gives rise to the noncommutativity matrix $\theta = (\theta^{\mu\nu})$ appearing in $[X^\mu(\tau), X^\nu(\tau)] = i\theta^{\mu\nu}$ [4]. The most general nontrivial 'magnetic' field $B = (B_{\mu\nu})$ has components

$$B_{12} = -B_{21} =: B_1 \,, \qquad B_{34} = -B_{43} =: B_2 \,, \tag{3.3}$$

[2] We use $\rho^0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\rho^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$, $(\eta^{\alpha\beta}) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma^0 = \tau$, $\sigma^1 = \sigma$. Recall that a Majorana spinor φ (in $1+1$ dimensions) has two components, and $\overline{\varphi} = \varphi^\dagger \rho^0$.

but depending on the dimension and orientation of the D-branes one or both parameters may be equal to zero. The inverse open-string metric and the noncommutativity parameters read

$$(G^{\mu\nu}) = \mathrm{diag}\left(\frac{\xi_1}{\xi_1^2+(2\pi\alpha' B_1)^2}, \frac{\xi_1}{\xi_1^2+(2\pi\alpha' B_1)^2}, \frac{-\xi_2}{\xi_2^2+(2\pi\alpha' B_2)^2}, \frac{-\xi_2}{\xi_2^2+(2\pi\alpha' B_2)^2}\right),$$
$$\theta^{12} = -\theta^{21} = -\frac{(2\pi\alpha')^2 B_1}{\xi_1^2+(2\pi\alpha' B_1)^2}, \qquad \theta^{34} = -\theta^{43} = -\frac{(2\pi\alpha')^2 B_2}{\xi_2^2+(2\pi\alpha' B_2)^2}. \tag{3.4}$$

Notice that for $B_2 = -B_1$ the background will be self-dual, and the action (2.4) will have $N = 4$ supersymmetry [43].

Finally, for the effective open-string coupling G_s, which is related to the closed-string coupling g_s via $G_s = g_s[\det G/\det(g + 2\pi\alpha' B)]^{1/2}$, we obtain

$$G_s = g_s\left[\left(1+\left(2\pi\frac{\alpha'}{\xi_1}B_1\right)^2\right)\left(1+\left(2\pi\frac{\alpha'}{\xi_2}B_2\right)^2\right)\right]^{1/2} = (2\pi)^{2-p}\,(\alpha')^{\frac{3-p}{2}}\,g^2_{\mathrm{YM}}, \tag{3.5}$$

including the relation to the Yang-Mills coupling g_{YM} on the Dp-brane [6].

D3-branes. Consider n coincident (kleinian) D3-branes with world-volume signature $(++--)$. Then the Seiberg-Witten limit consists of taking $\alpha'\to 0$ while sending $\xi_1 = \xi_2 = (\alpha')^2 \to 0$ (and therefore $g_{\mu\nu} \to 0$) so that G, G^{-1}, and θ remain finite. Let us now rescale coordinates,

$$x^1 \to 2\pi B_1 x^1, \qquad x^2 \to 2\pi B_1 x^2, \qquad x^3 \to 2\pi B_2 x^3, \qquad x^4 \to 2\pi B_2 x^4, \tag{3.6}$$

and denote them by the same letter x^μ. Then, the components of G and θ must be transformed as befits rank-two tensors. Explicitly, we have

$$(G^{\mu\nu}) \to (\eta^{\mu\nu}) = \mathrm{diag}(+1,+1,-1,-1),$$
$$\theta^{12} = -\theta^{21} \to -4\pi^2 B_1, \qquad \theta^{34} = -\theta^{43} \to -4\pi^2 B_2. \tag{3.7}$$

Thus, in these coordinates the commutative limit $\theta \to 0$ corresponds to $B \to 0$.

D2-branes. Next we have a look at n coincident (minkowskian) D2-branes with world volumes extending along x^1, x^2, x^4. In this case we take $\xi_1 = (\alpha')^2$ but $\xi_2 = 1$ and in the Seiberg-Witten limit send $\alpha' \to 0$. Our subsequent rescaling for this case,

$$x^1 \to 2\pi B_1 x^1, \qquad x^2 \to 2\pi B_1 x^2, \qquad x^3 \to x^3, \qquad x^4 \to x^4, \tag{3.8}$$

sends the inverse open-string metric and the noncommutativity parameter to

$$(G^{\mu\nu}) \to (\eta^{\mu\nu}) = \mathrm{diag}(+1,+1,-1,-1), \qquad \theta^{12} = -\theta^{21} \to -4\pi^2 B_1 =: \theta \tag{3.9}$$

(remembering that $\theta^{34} = 0$ in this case). It is obvious that the D2-brane results can be obtained from the D3-brane equations by simply setting $B_2 = 0$. Note that by considering euclidean D1-branes we arrive at the same formulae (3.9).

4. Effective field theories on D-brane world volumes

Eleven years ago it was discovered [11] that the open $N = 2$ fermionic string at tree level is identical to *self-dual* Yang-Mills field theory in $2 + 2$ dimensions. The complete absence of a massive physical spectrum ties in with the vanishing of all string amplitudes beyond three-point. In [12] it was shown that turning on a non-vanishing constant B-field background yields *noncommutative* self-dual Yang-Mills as the effective field theory in the target. We shall now extend our analysis to the case where the B-field is restricted to the world volume of lower-dimensional D-branes.

Apart from replacing $g \to G$ in the propagators (3.1), turning on a B-field merely multiplies each primitive open-string amplitude, obtained at $\theta = 0$ from a disk diagram with a fixed leg ordering $(1, 2, \ldots, M)$, by a factor of

$$\prod_{1\leq j<\ell\leq M} \exp\left\{-\frac{i}{2} k_{j\mu}\, \theta^{\mu\nu}\, k_{\ell\nu}\right\}, \tag{4.1}$$

where k_j denotes the momentum of the jth external leg. As delineated in [6] the effect of the phase (4.1) is subsumed in the effective world-volume field theory by replacing the ordinary product of fields with the Moyal (star) product.

The $N = 2$ string is no exception to this rule [12]: Its tree-level amplitudes on the space-time filling brane are reproduced by the noncommutative extension of the scalar field theory for a *prepotential* of *self-dual* $\mathrm{U}(n)$ gauge theory. The choice of prepotential is a matter of gauge. In fact, by choosing a suitable Lorentz frame we can always arrive at Leznov's gauge [44] or at Yang's gauge [45] and even scale the Yang-Mills coupling g_{YM} to unity [46]. Since the Leznov as well as the Yang gauge break $\mathrm{SO}(2, 2)$ invariance, the resulting scalar field theories and equations have a reduced global symmetry. In the following we shall employ both gauges and restrict to lower-dimensional D-branes by setting appropriate B-field components and coordinate dependencies to zero.

D3-branes. This is the maximal case already discussed in [12]. The cubic lagrangian for the $u(n)$-valued Leznov prepotential ϕ reads

$$L = \frac{1}{2}\eta^{\mu\nu}\, \mathrm{tr}\, \partial_\mu\phi\star\partial_\nu\phi + \frac{1}{3}\, \mathrm{tr}\, \phi\star[(\partial_2+\partial_4)\phi\star(\partial_1-\partial_3)\phi - (\partial_1-\partial_3)\phi\star(\partial_2+\partial_4)\phi]\,, \tag{4.2}$$

where 'tr' traces over the $u(n)$ algebra and fields are multiplied by means of the noncommutative star product

$$(f\star g)(x) = f(x) \exp\left\{\frac{i}{2} \overleftarrow{\partial}_\mu\, \theta^{\mu\nu}\, \overrightarrow{\partial}_\nu\right\} g(x)\,. \tag{4.3}$$

D2-branes. For the case of coincident D2-branes our reasoning leads to the lagrangian

$$L = \frac{1}{2}\eta^{ab}\, \mathrm{tr}\, \partial_a\phi\star\partial_b\phi + \frac{1}{3}\, \mathrm{tr}\, \phi\star[(\partial_2+\partial_4)\phi\star\partial_1\phi - \partial_1\phi\star(\partial_2+\partial_4)\phi]\,, \tag{4.4}$$

where $a, b, \ldots = 1, 2, 4$. We see that in accordance with the discussion above the lagrangian (4.4) can be obtained from (4.2) upon reducing from the D3-brane world volume to a D2-brane world volume by imposing $\partial_3 \phi = 0$. Our choice of B-field then implies (see (3.9)) that the time coordinate x^4 becomes commutative. From now on, we shall relabel our coordinates as follows,

$$x = x^1 \,, \qquad y = x^2 \,, \qquad t = -x^4 \,. \tag{4.5}$$

The lagrangian (4.4) produces a Leznov-type equation of motion,

$$\partial_x^2 \phi - \partial_u \partial_v \phi + \partial_v \phi \star \partial_x \phi - \partial_x \phi \star \partial_v \phi = 0 \,, \tag{4.6}$$

where we defined

$$u := \frac{1}{2}(t + y) \,, \qquad v := \frac{1}{2}(t - y) \,, \qquad \partial_u = \partial_t + \partial_y \,, \qquad \partial_v = \partial_t - \partial_y \,. \tag{4.7}$$

D1-branes. By considering open $N = 2$ strings interacting with n coincident D1-branes extending in the x-direction we obtain in the Seiberg-Witten limit (see section 3) an effective field theory on the D1-brane world volume $\mathbb{R}^{1,1}$, with the following equation of motion:

$$\partial_x^2 \phi - \partial_t^2 \phi + \partial_t \phi \star \partial_x \phi - \partial_x \phi \star \partial_t \phi = 0 \,. \tag{4.8}$$

Alternatively, for euclidean D1-branes the effective scalar field on $\mathbb{R}^{2,0}$ obeys

$$\partial_x^2 \phi + \partial_y^2 \phi + \partial_x \phi \star \partial_y \phi - \partial_y \phi \star \partial_x \phi = 0 \,. \tag{4.9}$$

It is easy to see that equations (4.8) and (4.9) can be obtained from equation (4.6) by imposing the conditions $\partial_y \phi = 0$ and $\partial_t \phi = 0$, respectively.

Operator formalism. In the remainder of this paper we concentrate on the noncommutative field theory (4.4) in $2 + 1$ dimensions, which can be obtained in a Seiberg-Witten limit of the open $N = 2$ string with n coincident D2-branes and a constant B-field background. It is well known that the star product, e.g. in (4.4), can be dropped by passing to noncommutative coordinates, $x^\mu \to \hat{x}^\mu$, satisfying $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$. In our D2-brane context we have $(t, x, y) \to (t, \hat{x}, \hat{y})$ where

$$[t, \hat{x}] = [t, \hat{y}] = 0 \,, \qquad [\hat{x}, \hat{y}] = i\theta \qquad \text{with } \theta = -4\pi^2 B_1 > 0 \,. \tag{4.10}$$

The metric $(\eta_{ab}) = \mathrm{diag}(+1, +1, -1)$ and the parameter θ are extracted from equation (3.9).

Using complex noncommutative coordinates $\hat{z} = \hat{x} + i\hat{y}$ and $\hat{\bar{z}} = \hat{x} - i\hat{y}$ we introduce creation and annihilation operators

$$a = \frac{1}{\sqrt{2\theta}}\, \hat{z} \qquad \text{and} \qquad a^\dagger = \frac{1}{\sqrt{2\theta}}\, \hat{\bar{z}} \qquad \text{so that} \quad [a, a^\dagger] = 1 \,. \tag{4.11}$$

They act on the Fock space $\mathcal{H}$ with an orthonormal basis $\{|n\rangle,\ n=0,1,2,\ldots\}$ such that

$$a\,|0\rangle = 0\,, \qquad a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{4.12}$$

Then, any function $f(t,z,\bar{z})$ can be related to an operator-valued function $\hat{f}(t)$ acting in $\mathcal{H}$, with the help of the Moyal-Weyl map (see e.g. [47, 48])

$$f(t,z,\bar{z}) \longrightarrow \hat{f}(t) = \int \frac{2i\,\mathrm{d}p\mathrm{d}\bar{p}}{(2\pi)^2}\ \tilde{f}(t,p,\bar{p})e^{-i\sqrt{2\theta}(\bar{p}a+pa^\dagger)}\,, \tag{4.13}$$

where $\tilde{f}(t,p,\bar{p})$ is the Fourier transform of $f(t,z,\bar{z})$ with respect to $(z,\bar{z})$. Under this map, we have $f\star g \to \hat{f}\,\hat{g}$ and

$$\int \mathrm{d}x\,\mathrm{d}y\,f \longrightarrow 2\pi\theta\,\mathrm{Tr}\,\hat{f} = 2\pi\theta\sum_{n\geq 0}\langle n|\hat{f}|n\rangle\,, \tag{4.14}$$

where 'Tr' signifies the trace over the Fock space $\mathcal{H}$. The operator formulation turns spatial derivatives into commutators,

$$\partial_x f \longrightarrow \frac{i}{\theta}\,[\hat{y},\hat{f}] =: \hat{\partial}_x\hat{f} \qquad \text{and} \qquad \partial_y f \longrightarrow -\frac{i}{\theta}\,[\hat{x},\hat{f}] =: \hat{\partial}_y\hat{f}\,. \tag{4.15}$$

For notational simplicity we will from now on omit the hats over the operators.

In the operator language, the Leznov-type equation (4.6) may be rewritten as follows,

$$\partial_x^2\phi - \partial_u\partial_v\phi + [\partial_v\phi\,,\,\partial_x\phi] = 0\,, \tag{4.16}$$

where the elements of the $u(n)$ matrices ϕ are t-dependent operators in the Fock space $\mathcal{H}$, and derivatives ∂_x, $\partial_u = \partial_t + \partial_y$ and $\partial_v = \partial_t - \partial_y$ are understood as in (4.15).

5. Noncommutative solitons in $2+1$ dimensions

In the previous sections we have laid the groundwork for the main part of this paper, which sets out to construct solitonic classical configurations for the integrable effective field theory of open $N=2$ strings on coincident noncommutative D2-branes. The crucial ingredient for solving the noncommutative field equations is the 'dressing method', which was invented to generate solutions for commutative integrable systems [38, 39, 40] and is easily extended to the noncommutative setup. The purpose of this section is, firstly, to present this important technique in its generality and adapt it to the noncommutative situation before, secondly, demonstrating its power by deriving one-soliton configurations for the U(1) and U(2) gauge groups and calculating their energies.

JHEP06(2001)011

Linear system. Let us consider the two linear equations

$$(\zeta\partial_x - \partial_u)\psi = A\,\psi \qquad \text{and} \qquad (\zeta\partial_v - \partial_x)\psi = B\,\psi\,, \tag{5.1}$$

which can be obtained from the Lax pair for the self-dual Yang-Mills equations in $\mathbb{R}^{2,2}$ [49] by gauge-fixing and imposing the condition $\partial_3\psi = 0$. Here, ψ depends on (t,x,y,ζ) or, equivalently, on (x,u,v,ζ) and is an $n{\times}n$ matrix whose elements act as operators in the Fock space $\mathcal{H}$. The matrices A and B are of the same type as ψ but do not depend on ζ. The spectral parameter ζ lies in the extended complex plane. The matrix ψ is subject to the following reality condition [13, 49]:

$$\psi(t,x,y,\zeta)\,[\psi(t,x,y,\bar{\zeta})]^\dagger = 1\,, \tag{5.2}$$

where '$\dagger$' is hermitean conjugation. We also impose on ψ the standard asymptotic conditions [49]

$$\psi(t,x,y,\zeta\to\infty) = 1 + \zeta^{-1}\phi(t,x,y) + O(\zeta^{-2})\,, \tag{5.3}$$

$$\psi(t,x,y,\zeta\to 0) = \Phi^{-1}(t,x,y) + O(\zeta)\,. \tag{5.4}$$

As we shall see in a moment, it is no coincidence that the letter ϕ in (5.3) is also used to denote the (operator-valued) Leznov field in (4.16).

Compatibility conditions. The compatibility conditions for the linear system of differential equations (5.1) are

$$\partial_x B - \partial_v A = 0\,, \tag{5.5}$$

$$\partial_x A - \partial_u B - [A,B] = 0\,. \tag{5.6}$$

The solution of (5.5) is

$$A = \partial_x\phi \qquad \text{and} \qquad B = \partial_v\phi\,, \tag{5.7}$$

where ϕ is the same matrix as in (5.3). By substituting (5.7) into (5.6) we reproduce the second-order equation (4.16), which identifies ϕ as the Leznov field.

Alternatively, one can first solve (5.6) via

$$A = \Phi^{-1}\partial_u\Phi \qquad \text{and} \qquad B = \Phi^{-1}\partial_x\Phi\,, \tag{5.8}$$

where Φ^{-1} is the matrix in (5.4). By inserting (5.8) into the remaining condition (5.5) we gain an equivalent (Yang-type) form of our field equation (4.16), namely

$$\partial_x(\Phi^{-1}\partial_x\Phi) - \partial_v(\Phi^{-1}\partial_u\Phi) = 0\,. \tag{5.9}$$

Returning to the original coordinates,

$$\partial_x(\Phi^{-1}\partial_x\Phi) + \partial_y(\Phi^{-1}\partial_y\Phi) - \partial_t(\Phi^{-1}\partial_t\Phi) + \partial_y(\Phi^{-1}\partial_t\Phi) - \partial_t(\Phi^{-1}\partial_y\Phi) = 0\,, \tag{5.10}$$

one notices that this equation is not $\mathrm{SO}(2,1)$ Lorentz invariant. However, it is the integrable equation of motion for a modified sigma model in $2+1$ dimensions [13]. Finally we remark that (5.7) and (5.8) also follow directly from (5.1) and the asymptotic forms (5.3) and (5.4).

JHEP06(2001)011

Solutions. In order to generate solutions of the equations (4.16) and (5.9) we employ the so-called 'dressing method' [38, 39, 40, 13]. More concretely, we look for solutions to the system (5.1) with (5.2) of the form

$$\psi(t,x,y,\zeta) = 1 + \sum_{k=1}^{m} \frac{\mu_k - \bar{\mu}_k}{\zeta - \mu_k}\, P_k(t,x,y)\,, \tag{5.11}$$

where the μ_k are complex constants (usually $\mathrm{Im}\,\mu_k < 0$ for all k) and the matrices P_k are independent of ζ. In the commutative case such ψ describe m *moving* solitons which scatter trivially and whose velocities are determined by the constants μ_k. Subjecting the ansatz (5.11) to the reality condition (5.2) we find that the P_k have to satisfy certain algebraic equations. An obvious solution to the latter is found by taking the P_k to be mutually orthogonal hermitean projectors:

$$P_k^\dagger = P_k\,, \qquad P_k^2 = P_k\,, \qquad P_{k_1}P_{k_2} = 0 \qquad \text{for } k_1 \neq k_2\,. \tag{5.12}$$

From (5.1) and (5.2) we learn that

$$A(t,x,y) = (\zeta\partial_x\psi - \partial_u\psi)(t,x,y,\zeta)\,[\psi(t,x,y,\bar{\zeta})]^\dagger\,, \tag{5.13}$$
$$B(t,x,y) = (\zeta\partial_v\psi - \partial_x\psi)(t,x,y,\zeta)\,[\psi(t,x,y,\bar{\zeta})]^\dagger\,. \tag{5.14}$$

The matrices A and B do not depend on ζ; therefore the poles at $\zeta = \mu_k$ on the r.h.s. of (5.13) and (5.14) have to be removable. This leads to the following differential equations for the projectors P_k:

$$(\mu_k\partial_x P_k - \partial_u P_k)\left(1 - \sum_{\ell=1}^{m}\frac{\mu_\ell - \bar{\mu}_\ell}{\mu_k - \bar{\mu}_\ell}P_\ell\right) = 0 = (\mu_k\partial_v P_k - \partial_x P_k)\left(1 - \sum_{\ell=1}^{m}\frac{\mu_\ell - \bar{\mu}_\ell}{\mu_k - \bar{\mu}_\ell}P_\ell\right). \tag{5.15}$$

If r_k is the matrix rank of P_k and $n{\geq}2$ we can always take P_k to be of the form

$$P_k = T_k\,(T_k^\dagger T_k)^{-1}\,T_k^\dagger \qquad \text{for } k = 1,\dots,m\,, \tag{5.16}$$

where the T_k are arbitrary $n\times r_k$ matrices subject to $T_k^\dagger T_\ell = 0$ for $k \neq \ell$. As before, the elements of $T_k, T_k^\dagger$ and P_k are Fock-space operators. Substituting (5.16) into (5.15) we find that the equations (5.15) hold if the $T_k^\dagger$ satisfy

$$\mu_k\partial_x T_k^\dagger - \partial_u T_k^\dagger = 0 \qquad \text{and} \qquad \mu_k\partial_v T_k^\dagger - \partial_x T_k^\dagger = 0\,. \tag{5.17}$$

The *general* solution of these equations is $T_k^\dagger = T_k^\dagger(\bar{w}_k)$, meaning the T_k are arbitrary functions of

$$w_k \;:=\; x + \bar{\mu}_k u + \bar{\mu}_k^{-1} v = x + \frac{1}{2}(\bar{\mu}_k - \bar{\mu}_k^{-1})\,y + \frac{1}{2}(\bar{\mu}_k + \bar{\mu}_k^{-1})\,t\,. \tag{5.18}$$

Hence, if the matrix elements of T_k are arbitrary holomorphic functions of the complex linear combination w_k of the coordinates t,x,y, then the projectors (5.16) satisfy the differential equations (5.15).

The sub-ansatz (5.12) is rather restrictive for describing *multi*-soliton configurations. Those are captured better by putting (cf. [40, 13])

$$P_k = \sum_{\ell=1}^{m} T_\ell \, \frac{\Gamma^{\ell k}}{\mu_k - \bar{\mu}_k} \, T_k^\dagger \qquad \text{for} \quad k = 1, \ldots, m \,, \tag{5.19}$$

where the $T_k(t,x,y)$ are $n \times r$ matrices, now unconstrained, and $\Gamma^{\ell k}(t,x,y)$ are $r \times r$ matrices with $r \geq 1$, so that

$$\psi(t,x,y,\zeta) = 1 + \sum_{k,\ell=1}^{m} \frac{1}{\zeta - \mu_k} \, T_\ell \, \Gamma^{\ell k} \, T_k^\dagger \,. \tag{5.20}$$

Substituting (5.20) into (5.2) we learn that the $\Gamma^{\ell k}$ must invert the $r \times r$ matrices

$$\widetilde{\Gamma}_{kp} = \frac{1}{\mu_k - \bar{\mu}_p} \, T_k^\dagger \, T_p \,, \qquad \text{i.e.} \qquad \sum_{k=1}^{m} \Gamma^{\ell k} \, \widetilde{\Gamma}_{kp} = \delta^\ell{}_p \,. \tag{5.21}$$

Demanding again vanishing residues for the poles at $\zeta = \mu_k$ on the r.h.s. of (5.13) and (5.14) we obtain the same differential equations as before, namely (5.17), but now for the $T_k^\dagger$ in the ansatz (5.20). Thus, again every collection of holomorphic matrix functions $T_k(w_k)$ provides a solution.

We observe that plugging the ansatz (5.11) into the formulae (5.3) and (5.4) reveals the solutions

$$\phi = \sum_{k=1}^{m} (\mu_k - \bar{\mu}_k) \, P_k \qquad \text{and} \qquad \Phi^{-1} = \Phi^\dagger = 1 - \sum_{k=1}^{m} \frac{\mu_k - \bar{\mu}_k}{\mu_k} \, P_k \tag{5.22}$$

of equations (4.16) and (5.9) in terms of projectors. In the case of (5.19) the solutions take the form

$$\phi = \sum_{k,\ell=1}^{m} T_\ell \, \Gamma^{\ell k} \, T_k^\dagger \qquad \text{and} \qquad \Phi^{-1} = \Phi^\dagger = 1 - \sum_{k,\ell=1}^{m} \frac{1}{\mu_k} \, T_\ell \, \Gamma^{\ell k} \, T_k^\dagger \,. \tag{5.23}$$

Note that formally the noncommutativity never entered our considerations.

Solitons. Formulae (5.22) and (5.23) represent two rather general classes of explicit solutions to our Leznov- and Yang-type field equations (4.16) and (5.9). As solutions of the reduced noncommutative self-dual Yang-Mills equations they are of BPS character. If we want to specialize to *solitons* (i.e. moving lumps of energy), we should impose the condition that the energy of such field configurations is finite. For this consideration we find it convenient to switch from the Leznov- to the Yang-type description, which makes use of the $\mathrm{U}(n)$-valued field $\Phi(t,x,y)$. The computation of the energy gets quite involved already in the commutative limit when the soliton configuration Φ is time-dependent [13, 50, 51, 52], and even more so for *multi*-soliton configurations. To keep the algebra manageable we concentrate on *one*-soliton solutions for the gauge groups U(1) and U(2).

Putting $m = 1$ and dropping the index we get $\Gamma = \frac{\mu-\bar{\mu}}{T^\dagger T}$, and the solutions (5.22) and (5.23) simplify to

$$\phi = (\mu - \bar{\mu})\, P \qquad \text{and} \qquad \Phi = 1 + \frac{\mu - \bar{\mu}}{\bar{\mu}}\, P\,. \tag{5.24}$$

It is assumed that μ is not real. Furthermore, we have $\partial_x = \partial_w + \partial_{\bar{w}}$ and

$$\begin{aligned} \partial_t &= \frac{1}{2}(\bar{\mu} + \bar{\mu}^{-1})\partial_w + \frac{1}{2}(\mu + \mu^{-1})\partial_{\bar{w}}\,, \\ \partial_y &= \frac{1}{2}(\bar{\mu} - \bar{\mu}^{-1})\partial_w + \frac{1}{2}(\mu - \mu^{-1})\partial_{\bar{w}}\,, \end{aligned} \tag{5.25}$$

so that (5.15) reduces to

$$(\partial_w P)\,(1-P) = 0 \qquad \Longrightarrow \qquad (1-P)\,\partial_{\bar{w}} P = 0\,. \tag{5.26}$$

Recall that our solutions Φ depend on t, x, y only through w and $\bar{w}$. It is therefore convenient to switch from $z, \bar{z}$ (as used in section 4) to the 'co-moving' coordinates $w, \bar{w}$. For the latter we have (in the operator formalism)

$$[w, \bar{w}] = \frac{i}{2}\theta\,(\mu - \bar{\mu} - \mu^{-1} + \bar{\mu}^{-1}) =: 2\beta > 0 \qquad \text{for} \quad \operatorname{Im}\mu < 0\,. \tag{5.27}$$

Notice that $w \equiv z$ and $\beta = \theta$ when $\mu = -i$ (*static* case). We introduce creation and annihilation operators $c^\dagger$ and c together with a Fock space $\mathcal{H}_t$ related to the 'co-moving' coordinates $w, \bar{w}$ via

$$\begin{aligned} c &= \frac{1}{\sqrt{2\beta}}\, w\,, \qquad & c^\dagger &= \frac{1}{\sqrt{2\beta}}\,\bar{w}\,, \qquad & |n\rangle_t &:= \frac{(c^\dagger)^n}{\sqrt{n!}}\,|0\rangle_t\,, \\ c\,|0\rangle_t &= 0\,, & c\,|n\rangle_t &= \sqrt{n}\,|n-1\rangle_t\,, & c^\dagger\,|n\rangle_t &= \sqrt{n+1}\,|n+1\rangle_t\,. \end{aligned} \tag{5.28}$$

Coordinate derivatives are then represented in the standard fashion as [47, 48]

$$\sqrt{2\beta}\,\partial_w \quad \longrightarrow \quad -[c^\dagger,\,\cdot\,]\,, \qquad \sqrt{2\beta}\,\partial_{\bar{w}} \quad \longrightarrow \quad [c,\,\cdot\,]\,. \tag{5.29}$$

In the case of commutative space-time coordinates there are no smooth soliton solutions with fields taking values in the abelian U(1) group or its Lie algebra since then equations (4.16) and (5.9) will be linear. For a noncommutative space, though, this is not the case, as has been widely observed in the literature. In order to solve (5.26) for the U(1) case, one may take as P simply the projector

$$P_0 = |0\rangle_t\langle 0|_t \tag{5.30}$$

onto a one-dimensional subspace of the Fock space $\mathcal{H}_t$. It is not difficult to see that P_0 satisfies the condition (5.26). Thus, configurations (5.24) with $P = P_0$ fulfil equations (4.16) and (5.9), respectively.

JHEP06(2001)011

Let us advance to a one-soliton solution for the group U(2). In this situation, we have $n = 2$, and our $r = 1$ projector P is constructed from a $2{\times}1$ matrix T. As a simple case we take

$$T = \begin{pmatrix} \lambda \\ c \end{pmatrix} \qquad \text{with } \lambda \in \mathbb{C} \setminus \{0\} \tag{5.31}$$

and create the projector

$$P = T \frac{1}{T^\dagger T} T^\dagger = \begin{pmatrix} \frac{\lambda\bar{\lambda}}{N+\lambda\bar{\lambda}} & \frac{\lambda}{N+\lambda\bar{\lambda}}\, c^\dagger \\ c\, \frac{\bar{\lambda}}{N+\lambda\bar{\lambda}} & c\, \frac{1}{N+\lambda\bar{\lambda}}\, c^\dagger \end{pmatrix} \qquad \text{where } N := c^\dagger c\,, \tag{5.32}$$

which satisfies (5.26). Formulae (5.24) now provide solutions of the field equations (4.16) and (5.9).

Energies. As in the commutative case [13], the model (4.4) has a conserved energy-momentum density,

$$\mathcal{P}_a = (\delta_a^b \delta_0^c - \frac{1}{2}\eta_{a0}\eta^{bc})\, \mathrm{tr}\, \partial_b \Phi^\dagger \star \partial_c \Phi\,, \tag{5.33}$$

which yields the energy functional

$$\begin{aligned} E &= \int \mathrm{d}x\, \mathrm{d}y\, \mathcal{P}_0 = \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{tr}\Big(\partial_t \Phi^\dagger \star \partial_t \Phi + \partial_x \Phi^\dagger \star \partial_x \Phi + \partial_y \Phi^\dagger \star \partial_y \Phi \Big) \\ &= \frac{1}{4} \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{tr}\Big[(\bar{\mu} + \bar{\mu}^{-1})^2\, \partial_w \Phi^\dagger \star \partial_w \Phi + (\mu + \mu^{-1})^2\, \partial_{\bar{w}} \Phi^\dagger \star \partial_{\bar{w}} \Phi + \\ &\qquad + (2 + \mu\bar{\mu} + \mu^{-1}\bar{\mu}^{-1})\, (\partial_w \Phi^\dagger \star \partial_{\bar{w}} \Phi + \partial_{\bar{w}} \Phi^\dagger \star \partial_w \Phi) \Big]\,, \end{aligned} \tag{5.34}$$

where the last expression is valid only for the one-soliton case. The integral is taken over the space-like plane $t = const$ and does not depend on t.

Using (5.28) and (5.29) and replacing $\int \mathrm{d}x\mathrm{d}y\, \mathrm{tr} \to 2\pi\theta\, \mathrm{Tr} \equiv 2\pi\theta\, \mathrm{Tr}_{\mathcal{H}_t}$, we can calculate the energy (5.34) of the abelian soliton $\Phi = 1 + \frac{\mu - \bar{\mu}}{\bar{\mu}} P_0$:

$$\begin{aligned} E(\mu)|_{\mathrm{U}(1)} &= \frac{1}{4} \frac{(\mu - \bar{\mu})(\bar{\mu} - \mu)(1 + \mu\bar{\mu})^2}{(\mu\bar{\mu})^2} \frac{4}{i(\mu - \bar{\mu} - \mu^{-1} + \bar{\mu}^{-1})} \times \\ &\quad \times 2\pi\beta\, \mathrm{Tr}(\partial_w P_0\, \partial_{\bar{w}} P_0 + \partial_{\bar{w}} P_0\, \partial_w P_0) \\ &= \frac{2\pi(\bar{\mu} - \mu)(1 + \mu\bar{\mu})}{i\, \mu\bar{\mu}} \cdot \underbrace{\mathrm{Tr}[c\,,\, P_0]\, [P_0\,,\, c^\dagger]}_{=1} = 8\pi\gamma \sin^2 \varphi\,. \end{aligned} \tag{5.35}$$

To arrive at the last expression, we have performed the polar decomposition

$$\mu = \alpha\, e^{-i\varphi} \qquad \text{and defined} \qquad \gamma = \frac{1 + \alpha^2}{2\alpha \sin \varphi}\,. \tag{5.36}$$

The velocity of the soliton in the xy plane is (cf. [13])

$$(v_x, v_y) = \Big(-\frac{2\alpha \cos \varphi}{1 + \alpha^2}, \frac{1 - \alpha^2}{1 + \alpha^2} \Big)\,, \qquad \text{so that} \qquad \gamma^{-1} = \sqrt{1 - v_x^2 - v_y^2}\,. \tag{5.37}$$

We repeat the computation for the U(2) soliton built with (5.32). To this end we differentiate $\Phi = 1 + \frac{\mu-\bar{\mu}}{\bar{\mu}} P$ with the help of (5.29) and make repeated use of the identities $f(N)c = cf(N-1)$ and $c^\dagger f(N) = f(N-1)\, c^\dagger$. Substituting into (5.34), we obtain

$$
\begin{aligned}
E(\mu)|_{\mathrm{U}(2)} &= \frac{2\pi(\bar{\mu}-\mu)(1+\mu\bar{\mu})}{i\,\mu\bar{\mu}} \cdot \mathrm{Tr}[c\,,\,P]\,[P\,,\,c^\dagger] \\
&= 8\pi\gamma\sin^2\varphi\,\mathrm{Tr}\left|\begin{pmatrix} c\,\frac{-\lambda\overline{\lambda}}{(N-1+\lambda\overline{\lambda})(N+\lambda\overline{\lambda})} & \frac{\lambda\,\lambda\overline{\lambda}}{(N+\lambda\overline{\lambda})(N+1+\lambda\overline{\lambda})} \\ c\,c\,\frac{-\overline{\lambda}}{(N-1+\lambda\overline{\lambda})(N+\lambda\overline{\lambda})} & c\,\frac{\lambda\overline{\lambda}}{(N+\lambda\overline{\lambda})(N+1+\lambda\overline{\lambda})} \end{pmatrix}\right|^2 \\
&= 8\pi\gamma\sin^2\varphi\,\mathrm{Tr}\left\{\frac{(\lambda\overline{\lambda})^2}{(N+\lambda\overline{\lambda})(N+1+\lambda\overline{\lambda})^2} + \frac{\lambda\overline{\lambda}\,N}{(N-1+\lambda\overline{\lambda})(N+\lambda\overline{\lambda})^2}\right\} \\
&= 8\pi\gamma\sin^2\varphi\sum_{n=0}^{\infty}\frac{\lambda\overline{\lambda}}{(n+\lambda\overline{\lambda})(n+1+\lambda\overline{\lambda})} = 8\pi\gamma\sin^2\varphi\,,
\end{aligned}
\tag{5.38}
$$

where some care must be taken for $|\lambda| = 1$. We see that the energies of the one-soliton solutions for the groups U(1) and U(2) are finite and depend on the velocity (5.37) of the solitons. Notice that the energies do not depend on θ and are identical to the commutative result for SU(2) [13].

In order to obtain a solution describing m solitons moving with the *same* velocity one should use in (5.31) two polynomials in c of degree m. More general multi-soliton solutions featuring *relative* motion of the individual lumps can be constructed by using m *different* matrices T_k and applying formulae (5.20)–(5.23) with (5.18). All these *moving* solitons have a brane-world interpretation: Since m-soliton solutions of an effective U(n) Yang-Mills-Higgs theory in $2+1$ dimensions correspond to m D0-branes inside n coincident D2-branes, we see that our solitons describe D0-branes *moving* inside the D2-branes.

6. Solitons in $1+1$ and instantons in $2+0$ dimensions

Consider the effective field theory on the D1-brane world volume briefly discussed in section 4. After changing field variables via $\partial_x\phi = \Phi^{-1}\partial_t\Phi$ and $\partial_t\phi = \Phi^{-1}\partial_x\Phi$ its equation of motion takes the form

$$\partial_x(\Phi^{-1}\partial_x\Phi) - \partial_t(\Phi^{-1}\partial_t\Phi) = 0\,. \tag{6.1}$$

This is the field equation of the standard sigma model in $1+1$ dimensions with the field Φ taking values in the group U(n).

Obviously our choice (3.3) of the B-field leads to commutative $\mathbb{R}^{1,1}$, where $f\star g = f\,g$, and there appears no star product in (6.1) (nor in (4.8)). In principle one can turn on a more general B-field background with nonzero '*electric*' components $B_{14} = -B_{41}$. In [53] this type of field has been considered in much detail, and it was

JHEP06(2001)011

shown that it does not admit a zero-slope limit which produces a field theory on a noncommutative space-time. Thus, open $N = 2$ strings only yield the sigma model on *commutative* $\mathbb{R}^{1,1}$. In order to derive solitonic solutions of its equations (4.8) and (6.1) one should simply drop the y dependence from all formulae of section 5 (i.e. impose $\partial_y = 0$ on all fields). No additional work is necessary here.

Let us finally have a look at the effective field theory on the *euclidean* D1-brane world volume. It is easy to see that after the field redefinition $\partial_x \phi = \Phi^{-1}\partial_y \Phi$ and $\partial_y \phi = -\Phi^{-1}\partial_x \Phi$ its equation of motion becomes

$$\partial_x(\Phi^{-1} \star \partial_x \Phi) + \partial_y(\Phi^{-1} \star \partial_y \Phi) = 0\,. \tag{6.2}$$

This equation describes the U(n) sigma model on the *noncommutative* euclidean space $\mathbb{R}^{2,0}$. It is interesting to note that these equations can be derived by reduction of the noncommutative self-dual Yang-Mills equations on $\mathbb{R}^{2,2}$ but not from their euclidean variant on $\mathbb{R}^{4,0}$ (cf. [54]). Solutions of (6.2) are easily obtained from the solutions in $2+1$ dimensions formulated in section 5. One should simply put $\mu_k = -i \ \forall k$ in all formulae. Then $w_k = x + iy = z$ and $\bar{w}_k = x - iy = \bar{z}$, i.e. instanton solutions of noncommutative two-dimensional sigma models arise as *static* solutions of the modified sigma model in $2+1$ dimensions. If in (5.24) we consider projectors of rank one, we get the $\mathbb{C}P^n$ chiral model on a noncommutative plane. Solutions of its field equation were studied in [24].

Acknowledgments

The work of O.L. and A.D.P. is partially supported by DFG under grant Le 838/7-1. B.S. is grateful to the 'Studienstiftung des deutschen Volkes' for support.

References

[1] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[2] Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031].

[3] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219].

[4] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[5] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Dirac quantization of open strings and noncommutativity in branes*, *Nucl. Phys.* **B 576** (2000) 578 [hep-th/9906161].

[6] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[7] N. Marcus, *A tour through $N = 2$ strings*, hep-th/9211059.

[8] J. Bischoff and O. Lechtenfeld, *Path-integral quantization of the (2,2) string*, *Int. J. Mod. Phys.* **A 12** (1997) 4933 [hep-th/9612218].

[9] O. Lechtenfeld, *Mathematics and physics of $N = 2$ strings*, hep-th/9912281.

[10] G. Chalmers, O. Lechtenfeld and B. Niemeyer, *$N = 2$ quantum string scattering*, *Nucl. Phys.* **B 591** (2000) 39 [hep-th/0007020].

[11] H. Ooguri and C. Vafa, *Geometry of $N = 2$ strings*, *Nucl. Phys.* **B 361** (1991) 469; *Self-duality and $N = 2$ string magic*, *Mod. Phys. Lett.* **A 5** (1990) 1389.

[12] O. Lechtenfeld, A.D. Popov and B. Spendig, *Open $N = 2$ strings in a B-field background and noncommutative self-dual Yang-Mills*, *Phys. Lett.* **B 507** (2001) 317 [hep-th/0012200].

[13] R.S. Ward, *Soliton solutions in an integrable chiral model in $2 + 1$ dimensions*, *J. Math. Phys.* **29** (1988) 386;
R.S. Ward, *Classical solutions of the chiral model, unitons and holomorphic vector bundles*, *Comm. Math. Phys.* **128** (1990) 319.

[14] G.V. Dunne, R. Jackiw, S.-Y. Pi and C.A. Trugenberger, *Self-dual Chern-Simons solitons and two-dimensional nonlinear equations*, *Phys. Rev.* **D 43** (1991) 1332, erratum *ibid.* **D45** (1992) 3012.

[15] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and $(2, 0)$ superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[16] K. Furuuchi, *Instantons on noncommutative R^4 and projection operators*, *Prog. Theor. Phys.* **103** (2000) 1043 [hep-th/9912047].

[17] P.-M. Ho, *Twisted bundle on noncommutative space and* U(1) *instanton*, hep-th/0003012.

[18] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[19] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204].

[20] D.J. Gross and N.A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204].

[21] K. Takasaki, *Anti-self-dual Yang-Mills equations on noncommutative spacetime*, *J. Geom. Phys.* **37** (2001) 291 [hep-th/0005194].

[22] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[23] D.P. Jatkar, G. Mandal and S.R. Wadia, *Nielsen-Olesen vortices in noncommutative abelian Higgs model*, *J. High Energy Phys.* **09** (2000) 018 [hep-th/0007078].

[24] B.-H. Lee, K. Lee and H.S. Yang, *The $CP(n)$ model on noncommutative plane*, *Phys. Lett.* **B 498** (2001) 277 [hep-th/0007140].

[25] D. Bak, *Exact multivortex solutions in noncommutative abelian Higgs theory*, *Phys. Lett.* **B 495** (2000) 251 [hep-th/0008204].

[26] B. Chen, H. Itoyama, T. Matsuo and K. Murakami, *Correspondence between noncommutative soliton and open string/D-brane system via gaussian damping factor*, hep-th/0010066.

[27] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, *Unstable solitons in noncommutative gauge theory*, *J. High Energy Phys.* **04** (2001) 001 [hep-th/0009142].

[28] J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [hep-th/0010060].

[29] K. Hashimoto, *Fluxons and exact BPS solitons in non-commutative gauge theory*, *J. High Energy Phys.* **12** (2000) 023 [hep-th/0010251].

[30] M. Hamanaka and S. Terashima, *On exact noncommutative BPS solitons*, *J. High Energy Phys.* **03** (2001) 034 [hep-th/0010221].

[31] D. Bak, S. U., K. Lee and J.-H. Park, *Noncommutative vortex solitons*, *Phys. Rev.* **D 63** (2001) 125010 [hep-th/0011099].

[32] G.S. Lozano, E.F. Moreno and F.A. Schaposnik, *Nielsen-Olesen vortices in noncommutative space*, *Phys. Lett.* **B 504** (2001) 117 [hep-th/0011205].

[33] G.S. Lozano, E.F. Moreno and F.A. Schaposnik, *Self-dual Chern-Simons solitons in noncommutative space*, *J. High Energy Phys.* **02** (2001) 036 [hep-th/0012266].

[34] M. Legaré, *Noncommutative generalized NS and super matrix KDV systems from a noncommutative version of (anti-)self-dual Yang-Mills equations*, hep-th/0012077.

[35] A. Khare and M.B. Paranjape, *Solitons in* 2+1 *dimensional non-commutative Maxwell Chern-Simons Higgs theories*, *J. High Energy Phys.* **04** (2001) 002 [hep-th/0102016].

[36] J.A. Harvey, *Komaba lectures on noncommutative solitons and D-branes*, hep-th/0102076.

[37] D. Bak, S.K. Kim, K.-S. Soh and J.H. Yee, *Noncommutative Chern-Simons solitons*, hep-th/0102137.

[38] V.E. Zakharov and A.V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique. (In russian)*, *Sov. Phys. JETP* **47** (1978) 1017.

[39] V.E. Zakharov and A.B. Shabat, *Integration of non-linear equations of mathematical physics by the method of inverse scattering, 2*, *Funct. Anal. Appl.* **13** (1979) 166.

[40] P. Forgács, Z. Horváth and L. Palla, *Solution generating technique for self-dual monopoles*, *Nucl. Phys.* **B 229** (1983) 77.

[41] P. Haggi-Mani, U. Lindström and M. Zabzine, *Boundary conditions, supersymmetry and a-field coupling for an open string in a b-field background*, *Phys. Lett.* **B 483** (2000) 443 [hep-th/0004061].

[42] B. Zumino, *Supersymmetry and Kähler manifolds*, *Phys. Lett.* **B 87** (1979) 203.

[43] L. Alvarez-Gaumé and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Comm. Math. Phys.* **80** (1981) 443.

[44] A.N. Leznov, *On equivalence of four-dimensional self-duality equations to continual analog of the main chiral field problem. (In russian)*, *Theor. Math. Phys.* **73** (1987) 1233.

[45] C.N. Yang, *Condition of self-duality for* SU(2) *gauge fields on euclidean four-dimensional space*, *Phys. Rev. Lett.* **38** (1977) 1377.

[46] O. Lechtenfeld and W. Siegel, $N = 2$ *worldsheet instantons yield cubic self-dual Yang-Mills*, *Phys. Lett.* **B 405** (1997) 49 [hep-th/9704076].

[47] L. Alvarez-Gaume and S.R. Wadia, *Gauge theory on a quantum phase space*, *Phys. Lett.* **B 501** (2001) 319 [hep-th/0006219].

[48] D.J. Gross and N.A. Nekrasov, *Solitons in noncommutative gauge theory*, *J. High Energy Phys.* **03** (2001) 044 [hep-th/0010090].

[49] T.A. Ivanova and O. Lechtenfeld, *Hidden symmetries of the open* $N = 2$ *string*, *Int. J. Mod. Phys.* **A 16** (2001) 303 [hep-th/0007049].

[50] P.M. Sutcliffe, *Nontrivial soliton scattering in an integrable chiral model in (2+1)-dimensions*, *J. Math. Phys.* **33** (1992) 2269; *Yang-Mills Higgs solitons in (2+1)-dimensions*, *Phys. Rev.* **D 47** (1993) 5470.

[51] T. Ioannidou, *Soliton solutions and nontrivial scattering in an integrable chiral model in (2+1) dimensions*, *J. Math. Phys.* **37** (1996) 3422 [hep-th/9604126].

[52] T. Ioannidou and W.J. Zakrzewski, *Solutions of the modified chiral model in (2+1) dimensions*, *J. Math. Phys.* **39** (1998) 2693 [hep-th/9802122].

[53] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *J. High Energy Phys.* **06** (2000) 021 [hep-th/0005040].

[54] T.A. Ivanova and A.D. Popov, *Self-dual Yang-Mills fields and Nahm's equations*, *Lett. Math. Phys.* **23** (1991) 29; *Self-dual Yang-Mills fields in* $D = 4$ *and integrable systems in* $1 \leq D \leq 3$, *Theor. Math. Phys.* **102** (1995) 280; *Some new integrable equations from the self-dual Yang-Mills equations*, *Phys. Lett.* **A 205** (1995) 158 [hep-th/9508129].